COSC 462

Finite Difference Methods

Piotr Luszczek

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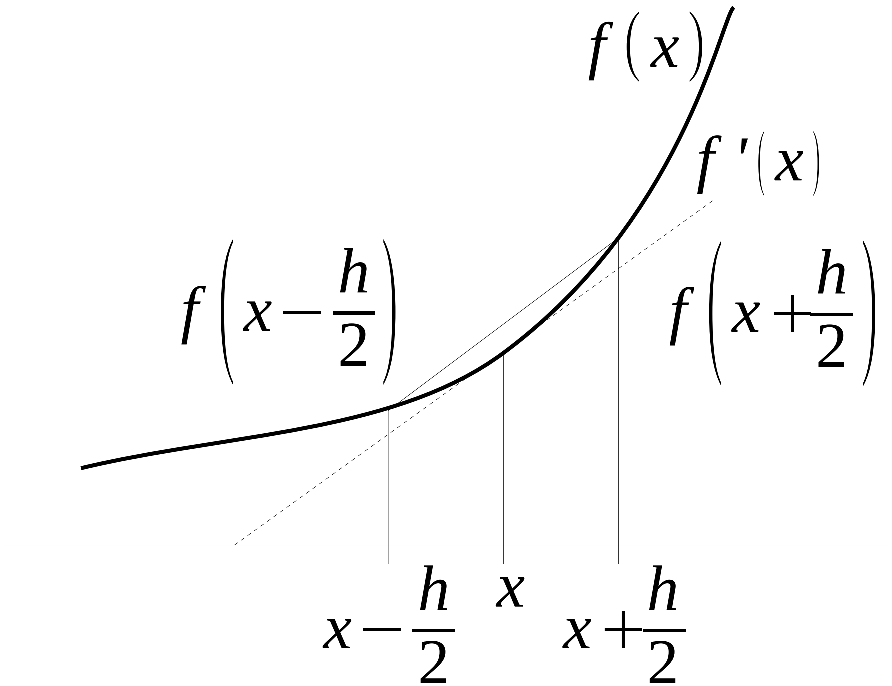
Finite Difference Methods and PDEs

- Finite Difference Methods are commonly used to solve PDEs
- PDEs are used in many applications
  - Computational Fluid Dynamics
    - Water and gas flow
    - Multi-scale models
      - Weather prediction
  - Structural mechanics
    - Deformations of rigid structures
  - Wave propagation
    - Acoustics

Approximating Derivative with Finite Difference



$$f'(x)=\frac{f(x+h/2)-f(x-h/2)}{h}+O(h^2)$$

$$f''(x)=\frac{f'(x+h/2)-f'(x-h/2)}{h}+O(h^2)$$

$$f''(x)\approx\frac{f(x+h)-2f(x)+f(x-h)}{h^2}$$

- The formulas above assume that:
  - function f() is continuous, and
  - so is its derivative f'()

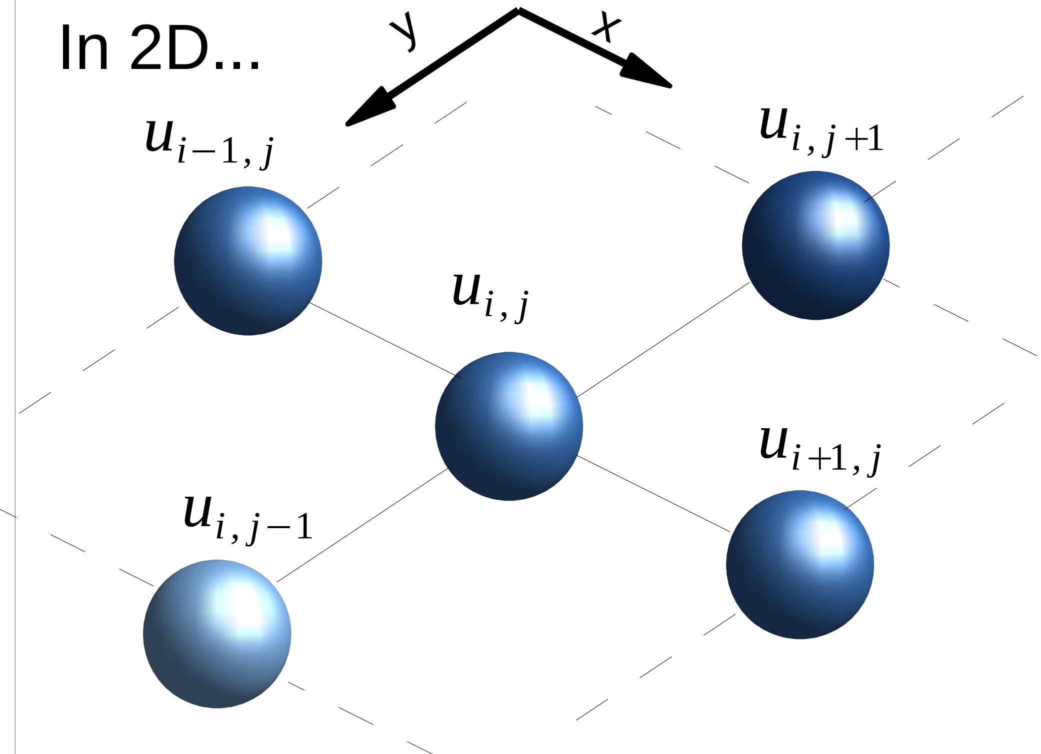
Sample PDE: Poisson Equation

- Poisson equation has a simple form in 2D
  - $u_{xx}+u_{yy}=f(x,y)$
- Applications include
  - Electricity
  - Magnetism
  - Gravity
  - Heat distribution
  - Fluid flow
  - Torsion
- When f(x,y)=0 we call it Laplace equation

$$u_{xx}=\frac{\partial^2 u}{\partial x \partial x}\approx\frac{u(x+h,y)-2u(x,y)+u(x-h,y)}{h^2}$$

Mapping Formulas to Geometry

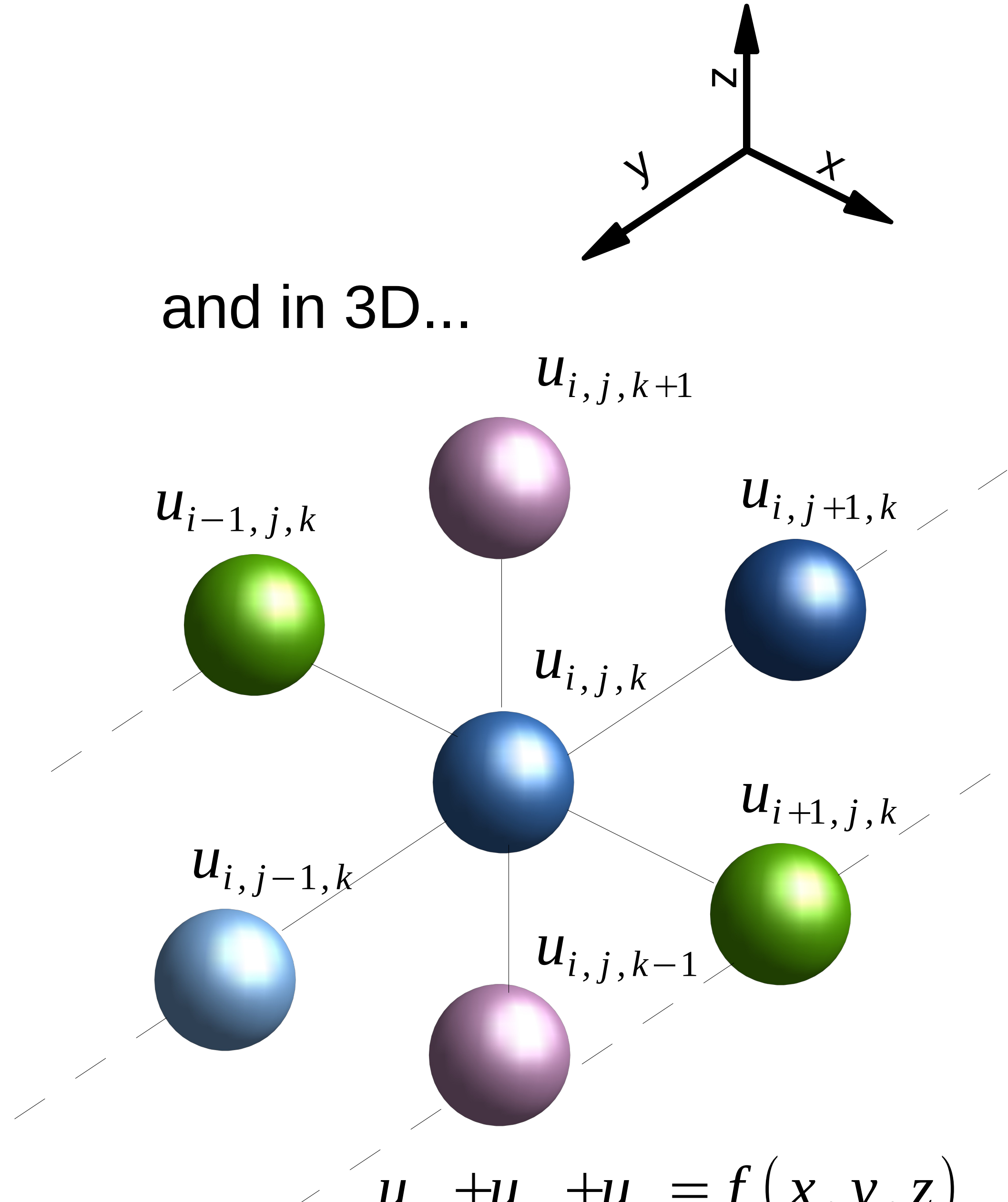
In 2D...



$$u_{xx} \approx \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

$$u_{yy} \approx \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2}$$

and in 3D...



$$u_{xx} + u_{yy} + u_{zz} = f(x,y,z)$$

Iterating Towards Steady-State

Start with $u_{i,j}$ estimates →

$$\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2}+\frac{u_{i,j-1}-2u_{i,j}+u_{i,j+1}}{h^2}=f_{i,j}$$

↓

$$\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2}+\frac{u_{i,j-1}-2u_{i,j}+u_{i,j+1}}{h^2}=f_{i,j}$$

↓

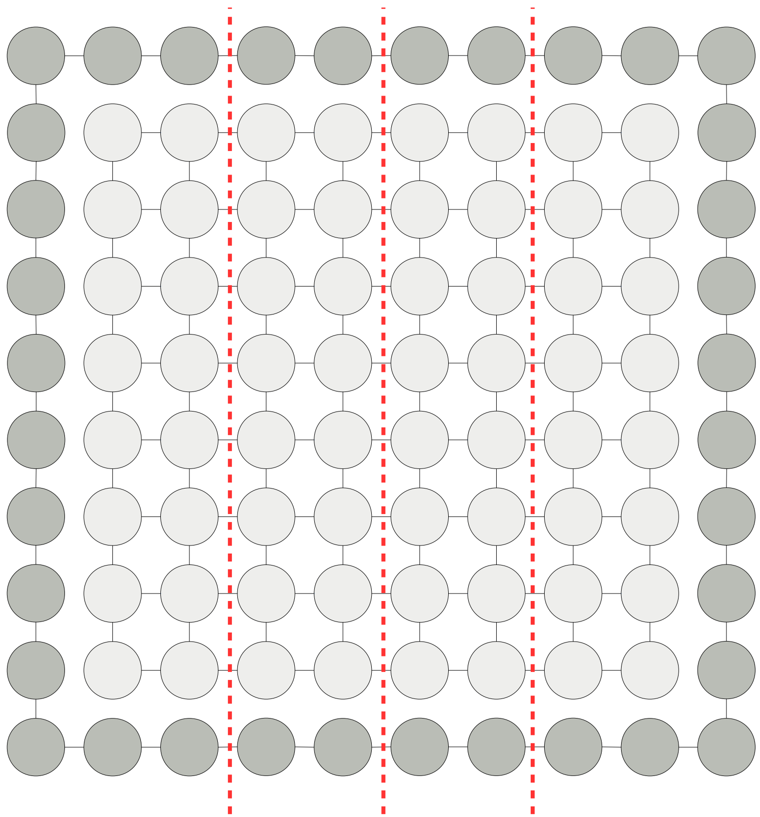
$$\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2}+\frac{u_{i,j-1}-2u_{i,j}+u_{i,j+1}}{h^2}=f_{i,j}$$

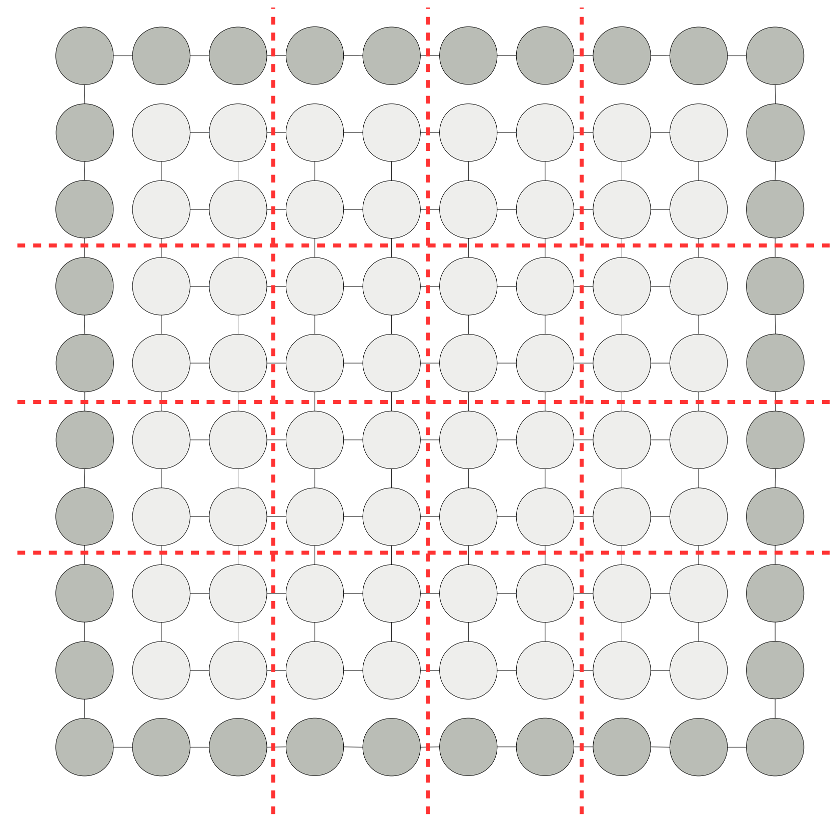
↓

$$\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2}+\frac{u_{i,j-1}-2u_{i,j}+u_{i,j+1}}{h^2}=f_{i,j}$$

Steady-state with final values of $u_{i,j}$ ←

Meshes: Partitioning and Agglomeration





Computation:

(N^2/P)

Communication (N by N mesh):

(N)

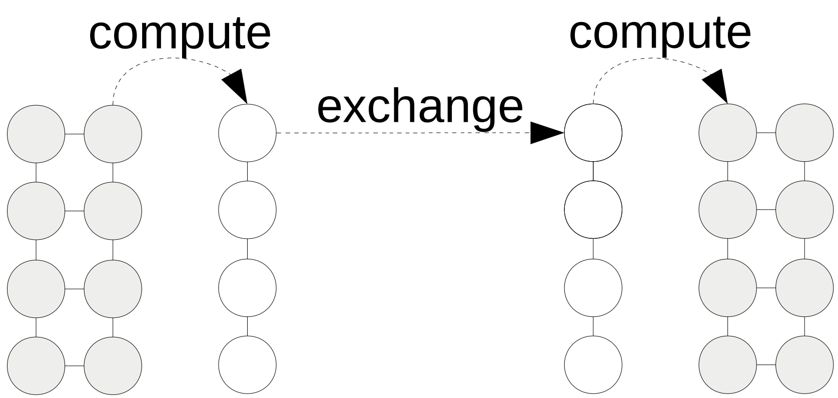
Computation:

(N/ P * N/ P) = (N^2/P)

Communication (N by N mesh):

(N/ P)

Implementation: Ghost Cells

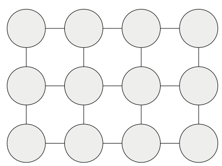


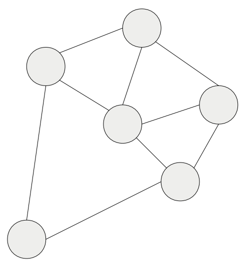
1. Compute on local cells
2. Compute on ghost cells
3. Exchange ghost cells
4. If not converged GOTO 1

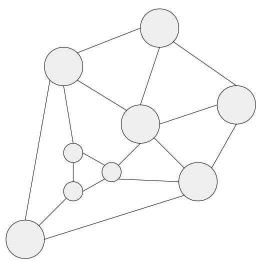
} This is usually combined in a clever implementation
} Communication is local

Details: Divisibility, Numerics, Mesh Refinement

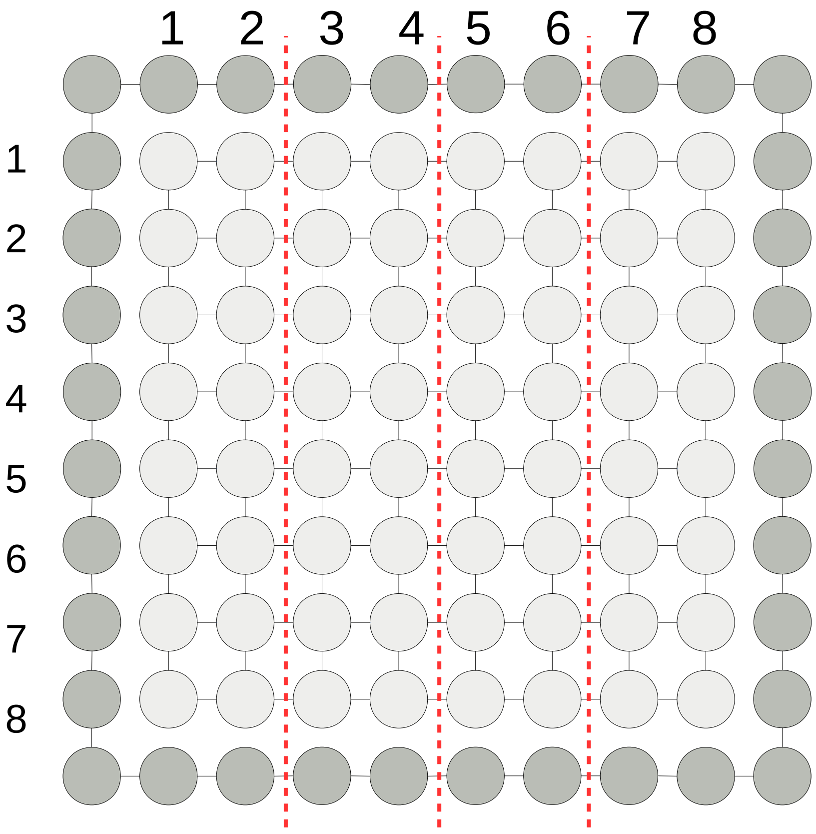
- Divisibility
  - More complex math (no simple way to pad to N+k)
    - We have to tolerate slight imbalance
  - Still want square processor grid
    - Might need to leave processors off for good prime factors
- Numerical issues
  - Convergence is a more complicated math problem
    - Need continuous boundary conditions etc.
  - More complicated PDEs and local solvers are a necessity
- Mesh structure
  - It does not always make sense to have uniform mesh
  - The mesh might change as computation proceeds







Mesh and Its Adjacency Matrix



	1,1	2,1	3,1	4,1	5,1	6,1,	7,1	8,1	1,2	2,2	3,2
1,1	-4	1	0	0	0	0	0	0	1	0	0
2,1	1	-4	1	0	0	0	0	0	0	1	0
3,1	0	1	-4	1	0	0	0	0	0	0	1
4,1	0	0	1	-4	1	0	0	0	0	0	0

Natural ordering
(other orderings possible:
red-black, nested dissection,
Cuthill-McKee, ...)

Adjacency matrix is sparse: